A Topological Account of the 2-adic Solenoid

Freda Shi and [Anonymous]

November 8 2023

1 Introductions and Motivations

In this paper, we provide an account of the homeomorphism between the Smale-Williams attractor and the 2-adic solenoid. The notion of inverse systems is first used to develop a topological notion of the 2-adic solenoid. The Smale-Williams attractor is then explicated and then the groundwork needed to prove homeomorphism between the solenoid and the attractor is laid. From there, we prove the homeomorphism between the Smale-Williams attractor and the the 2-adic solenoid. In proving the homeomorphism, readers encountering the 2-adic numbers for the first time are given a means to conceive of the 2-adic integers as they would appear embedded in $\mathbb{R}^3$. This visualization is both relatively intuitive and illuminates possibilities of studying the 2-adics via dynamic, topological, and algebraic means.

2 $\mathbb{S}_2$ is Homeomorphic to the Smale-Williams Attractor

Definition 2.1. An *inverse system* $(X_n, \varphi_n)_{n\geq 0}$ is a sequence of sets X_n and transition maps $\varphi_n : X_{n+1} \to X_n$. A set X with maps $\psi_n : X \to X_n$ such that $\psi_n = \varphi_n \circ \psi_{n+1}$ for $n \geq 0$ is the *inverse limit* (X, ψ_n) of the inverse system $(X_n, \varphi_n)_{n\geq 0}$ if for each set A and mappings $f_n : A \to X_n$ satisfying $f_n : \varphi_n \circ f_{n+1}$, there is a unique factorization $f$of f_n such that for $n \geq 0$,

$$f_n = \psi_n \circ f : A \to X \to X_n.$$

The inverse limit is denoted as $X = \varprojlim X_n$.

Definition 2.2. The 2-adic solenoid $\mathbb{S}_2$ is the inverse limit $\mathbb{S}_2 = \varprojlim \mathbb{R}/2^n\mathbb{Z}$ of the inverse system $(\mathbb{R}/2^n\mathbb{Z}, \varphi_n)_{n\geq 0}$.

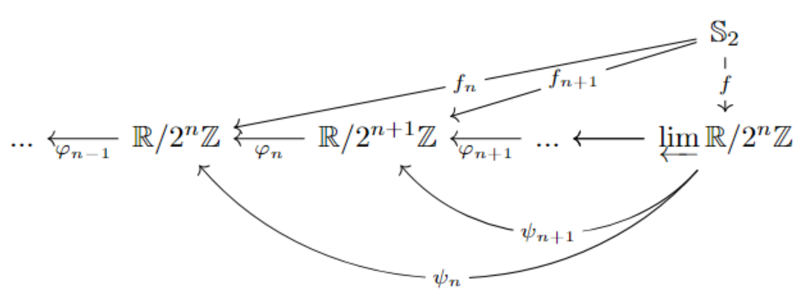


Figure 1: Visual depiction of $\mathbb{S}_2 = \varprojlim \mathbb{R}/2^n\mathbb{Z}$.

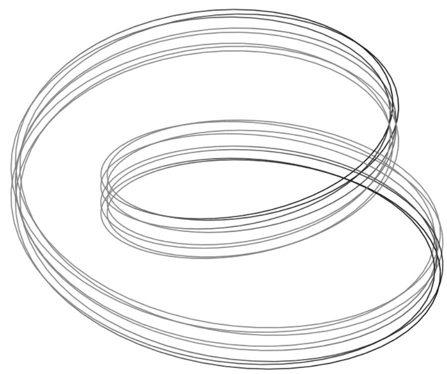


Figure 2: $\mathbb{S}_2$ in $\mathbb{R}^3$

.

Definition 2.3. On the solid torus $M = S^1 \times D^2$, where S^1 is 1-dimensional sphere and D^2 is the unit disc in $\mathbb{R}^2$, we define coordinates (φ, x, y) such that $\varphi \in S^1$ and $x, y \in D^2$. Using these coordinates, we define the map the *Smale-Williams attractor* as

$$T : M \to M, f(\varphi, x, y) = (2\varphi, \tfrac{1}{10}x + \tfrac{1}{2}\cos\varphi, \tfrac{1}{10}y + \tfrac{1}{2}\sin\varphi).$$

Lemma 2.4. The Smale-Williams attractor is well-defined and injective.

Proof. To verify that the map is well-defined, that is $T(M) \subset M$, we check

$$\begin{aligned}(\frac{1}{10}x + \frac{1}{2}\cos\varphi)^2 + (\frac{1}{10}y + \frac{1}{2}\sin\varphi)^2 =\\ \frac{1}{100}(x^2+y^2) + \frac{1}{10}(x\cos\varphi + y\sin\varphi) + \frac{1}{4}(\cos^2\varphi + \sin^2\varphi)\\ \leq \frac{1}{100} + \frac{2}{10} + \frac{1}{4} < 1.\end{aligned}$$

Thus, T(M) is contained in the interior of M.
Next, we check that T is injective. Suppose $T(\varphi_1, x_1, y_1) = T(\varphi_2, x_2, y_2)$. Then

$$\begin{aligned}2\varphi_1 &= 2\varphi_2,\\ \frac{1}{10}x_1 + \frac{1}{2}\cos\varphi_1 &= \frac{1}{10}x_2 + \frac{1}{2}\cos\varphi_2,\\ \frac{1}{10}y_1 + \frac{1}{2}\sin\varphi_1 &= \frac{1}{10}y_2 + \frac{1}{2}\sin\varphi_2.\end{aligned}$$

If $\varphi_1 = \varphi_2$ we see that $x_1 = x_2$ and $y_1 = y_2$. If $\varphi_1 = \varphi_2 + \pi$ then

$$\begin{aligned}\frac{1}{10}x_1 + \frac{1}{2}\cos\varphi_1 &= \frac{1}{10}x_2 - \frac{1}{2}\cos\varphi_1,\\ \frac{1}{10}y_1 + \frac{1}{2}\sin\varphi_1 &= \frac{1}{10}y_2 - \frac{1}{2}\sin\varphi_1\end{aligned}$$

or

$$\frac{1}{10}(x_2 - x_1) = \cos\varphi_1 and \frac{1}{10}(y_2 - y_1) = \sin\varphi_1$$

which implies

$$(x_2 - x_1)^2 + (y^2 - y_1)^2 = 100.$$

Since the left-hand side is bound by 8, this is impossible. □

Lemma 2.5. $T^n(S^1 \times D^2) \cap D_\theta{}^2$ is a union of 2^{n-1} discs. Each disc is contained in a unique disc of $T^{n-1}(S^1 \times D^2) \cap D_\theta{}^2$.

Proof. From Lemma 2.5, we see that any cross section $C = \theta \times D^2$ of M, the image of $f(M)$ will intersect C in two disjoint disc of radius $\frac{1}{10}$. $C \cap T(M)$ can be written as $T(C_1) \cup T(C_2)$, where C_1 and C_2 are two cross sections. Considering $T^2(M)$, from Lemma 2.5 we know that $f^2(M) \subset T(M)$. Moreover, $C \cap T^2(M) = T(C_1 \cap T(M)) \cup T(C_2 \cap T(M))$, where C_1 and C_2 are as before. Thus, $C \cap T^2(M)$ consists of four little discs, two each inside the discs $T(C_1)$ and $T(C_2)$.

Thus, $f^2(M)$ winds around M four times. As we continue to consider successive images $T^l(M)$ we thus find that $T^{l+1}(M)$ consists of 2^{l+1} discs, two each inside the discs of $C \cap T^l(M)$. □

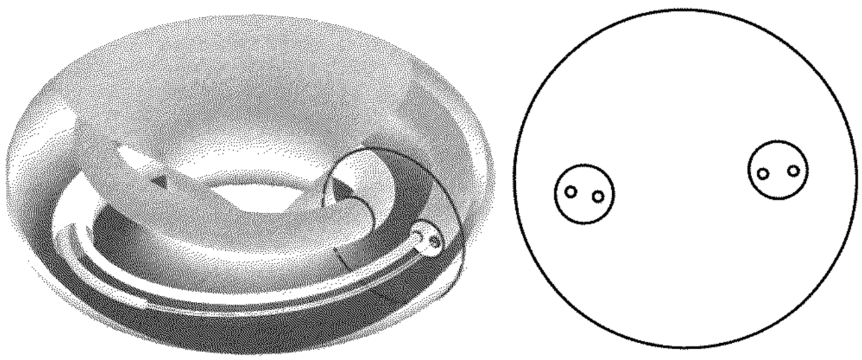


Figure 3: The Smale-Williams solenoid and one cross-section.

Definition 2.6. From the previous two lemmas, we define the maximal invariant set of T as $\lambda := M \cap f(M) \cap f^2(M)... = \cap_{l \in \mathbb{N}_0} f^l(M)$.

Lemma 2.7. $T|_\lambda : \lambda \to \lambda$ is a homeomorphism.

Proof. As each function component of T is continuous, so is T and $T|_\lambda$. λ is closed as any open ball in $S^1 \times D^2$ will fail to be contained in λ. Hence, λ is compact. Since λ is the maximal invariant set of T, $T^{-1}|_\lambda$ exists and is well-defined on λ. □

Lemma 2.8. There is a homeomorphism γ_1 between S^1 and the interval [0,1] where 0 and 1 are identified be the same endpoint.

Proof. Define $\gamma_1 : [0,1] \to S^1$, $\gamma_1(t) = (\cos(2\pi t), \sin(2\pi t))$.

We begin by showing that γ_1 is one-to one. Suppose $\gamma_1(a) = \gamma_1(b)$. This means that $(\cos(2\pi a), \sin(2\pi a)) = (\cos(2\pi b), \cos(2\pi b))$. As cosine and sine have periods of 2π, a and b must either be the same number or differ by an integer. If a, b $\in$ (0,1), $a - b \notin \mathbb{Z}$ thus a = b. If a, b $\in$ 0,1, a = b by definition. Thus, γ_1 is one-to-one.

Next, we will show that γ_1 is onto. Let a $\in S^1$ be given. Then a can be expressed as $(\cos(2\pi x), \sin(2\pi x))$, where x $\in \mathbb{R}$. Take $x' = x - \lfloor x \rfloor$. As cosine and sine have periods of 2π, $(\cos(2\pi x), \sin(2\pi x)) = (\cos(2\pi x'), \sin(2\pi x'))$. As x' $\in$ [0,1], we have shown that γ_1 is one-to-one.

Thus, γ_1 is bijection. Furthermore, γ_1 is also continuous as sine and cosine are continuous. As γ_1 is a continuous bijection, γ_1 is a homeomorphism.

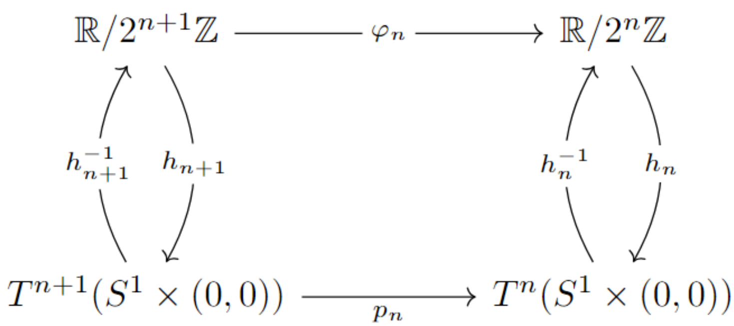
□

Lemma 2.9. There exists a parameterization of $T^n(S^1 \times (0,0))\gamma_n : [0, 2^n] \to T^n(S^1 \times (0,0))$ such that $\gamma_1'(\lambda) = 1 \forall t \iff \gamma_1(t) = t$. Define $h_n(t) = \gamma(t)$.

Proof. By the definition of T, T, is a continuous deformation of $S^1 \times D^2$. Hence, T is also a continuous deformation of $S^1 \times (0,0)$. $S^1 \times (0,0) \cong S^1$, which is parameterized by $[0, a]$ for any a $\in \mathbb{R}$. Since S^1 is parameterizable, so is $S^1 \times (0,0)$ and so is $T(S^1 \times (0,0))$ and so is $T^n(S^1 \times (0,0))$ for any $n \in \mathbb{N}$. Given a parameterization $\gamma_n : [0, 2^n] \to T^n$, define h_n to be γ_n. □

Lemma 2.10. There is a homeomorphism between the sequences $\{s_n\}$ in $\mathbb{S}_2$ with sequences $\{\lambda_n\}$, where each $\lambda_i \in T^i(S^1 \times (0,0))$ and $p_i(\lambda_{i+1}) = \lambda_i$.

Proof. First, we will show that the diagram below commutes.



Consider $\varphi_n(s_{n+1}) = s_{n+1} \bmod 2^n$. $\varphi(s_{n+1}) = \varphi(s_{n+1} + 2^n)$. As $\mathbb{R}/2^{n+1}\mathbb{Z} \cong \mathbb{R}/2^n\mathbb{Z} \cong S^1 \cong S^1 \times (0,0)$, this map sends points π radians apart from each other on the unit circle to the same point in $\mathbb{R}/2^{n+1}\mathbb{Z}$ and sends these points uniquely to s_n.

Consider $p_n(\lambda_{n+1})$. By lemma 2.6, every disc in $T^n(S^1 \times (0,0)) \cap D^2_\theta$ contains exactly two points $\lambda_{n+1}, \lambda_{n+1}$ of $T^{n+1}(S^1 \times (0,0))$ in it. p_n maps these two points to the single of of $T^n(S^1 \times (0,0))$ contained in the same disc as them. These two points, as they are contained in $T^n(\theta_k \times D^2)$ for some $\theta_k \in S^1$, are contained similarly in the same sequence of $n+1$ concentric discs. By this fact, $T^{-n}(\lambda_{n+1})$ and $T^{-n}(\lambda_{n+1})$ must have been the only two unique points of $T^1(S^1 \times (0,0))$ contained in some given discs D^2_θ. The only two points θ_1, θ_2 to be mapped to the same disc D^2_θ after one iteration of T are $\frac{\theta}{2}$ and $\frac{\theta}{2} + \pi$. Hence, p_n maps points of $S^1 \times (0,0)$ which were π radians apart on the unit circle to the same λ_n. Hence, the diagram commutes.

As the diagram commutes, given any sequence $\{s_n\}$ in $\mathbb{S}_2$, the sequence $\{\lambda_n\} = \{h_n(s_n))\}$ is a compatible sequence with respect to transition maps p_n. Denote the map $h' : \{s_n\} \mapsto \{\lambda_n\}$. As this map is given by the infinite product of homeomorphisms, $\bigcap_{i=0}^{\infty}(h_i)$, h' itself is a homeomorphism.

□

Theorem 2.11. $\mathbb{S}_2$ is homeomorphic to Λ.

Proof. Given a compatible sequence of points $\{\lambda_n\}$, each λ_i is contained uniquely in a disc $T^i(D^2_{\theta_i})$. Hence, to each $\{\lambda_n\}$ we may associate a sequence of concentric discs. These sequences are unique and each one converges to a point λ, as D^2_{i+1} is $\frac{1}{10}$ the radius of D_i. As this point is contained in $\bigcap_{i=0}^{\infty} T^i(M)$, given any $\lambda \in \Lambda$, λ is contained uniquely in a series of concentric discs $\{D^2_n\}$, $D^2_1 = T^1(\theta_i \times D^2)$. Each sequence of discs corresponds bijectively to a compatible sequence $\{\lambda_n\}$. Furthermore, each $\{\lambda_n\} = \{h_n(s_n)\}$ converges to the $\lambda \in \Lambda$ associated to it. Hence, $\mathbb{S}_2$ is homeomorphic to Λ. □